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# A FAMILY OF COUNTABLY COMPACT $P_*$-HYPERGROUPS

BY

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ABSTRACT. An infinite compact group is necessarily uncountable, by the Baire category theorem. A compact $P_*$-hypergroup, in which the product of two points is a probability measure, is much like a compact group, having an everywhere supported invariant measure, an orthogonal system of characters which span the continuous functions in the uniform topology, and a multiplicative semigroup of positive-definite functions. It is remarkable that a compact $P_*$-hypergroup can be countably infinite. In this paper a family of such hypergroups, which include the algebra of measures on the $p$-adic integers which are invariant under the action of the units (for $p = 2, 3, 5, \cdots$) is presented. This is an example of the symmetrization technique. It is possible to give a nice characterization of the Fourier algebra in terms of a bounded-variation condition, which shows that the usual Banach algebra questions about the Fourier algebra, such as spectral synthesis, and Helson sets have easily determinable answers. Helson sets are finite, each closed set is a set of synthesis, the maximal ideal space is exactly the underlying hypergroup, and the functions that operate are exactly the Lip 1 functions.

**Introduction.** An infinite compact group is necessarily uncountable, by the Baire category theorem. A compact $P_*$-hypergroup, in which the product of two points is a probability measure, is much like a compact group, having an everywhere supported invariant measure, an orthogonal system of characters which span the continuous functions in the uniform topology, and a multiplicative semigroup of positive-definite functions. It is remarkable that a compact $P_*$-hypergroup can be countably infinite. In this paper we present a family of such hypergroups, which include the algebra of measures on the $p$-adic integers which are invariant under the action of the units (for $p = 2, 3, 5, \cdots$). This is an example of the symmetrization technique (see [3]). It is possible to give a nice characterization of the Fourier algebra in terms of a bounded-variation condition, which shows that exactly the Lipschitz functions operate in the Fourier algebra.

The first chapter gives the definitions, and construction of the example, a family of $P_*$-hypergroups $H_a$, for $0 < a \leq \frac{1}{2}$, each of which has the topological

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structure of the one-point compactification of the nonnegative integers. The case $H_{1/p}$, $p$ prime, is the set of equivalence classes of the $p$-adic integers modulo the group of units (under multiplication). §5 shows for $0 < a < b \leq ½$ that natural homomorphisms exist from $H_a$ to $H_b$. §6 shows that both the partial summation and the Poisson kernels are positive.

In Chapter 2 characterizations of the forward and inverse Fourier transforms of various $L^p$-spaces are presented. These were developed by R. Spector [5] in the $p$-adic case. The reader will note the significant difference between this situation and the classical Fourier series case.

Finally in Chapter 3 we show that the usual Banach algebra questions about the Fourier algebra, such as spectral synthesis, and Helson sets have easily determinable answers. Helson sets are finite, each closed set is a set of synthesis, the maximal ideal space is exactly the underlying hypergroup, and the functions that operate are exactly the Lip 1 functions.

## CHAPTER I

1. The basic theory of hypergroups has been developed by Dunkl in [1]. A hypergroup $H$ is a compact space on which the space $M(H)$ of (finite) regular Borel measures is a commutative Banach algebra under its natural norm, possessing a multiplication (denoted by $*$), and such that the space $M_p(H)$ of probability measures is a compact commutative topological (jointly continuous multiplication) semigroup with unit $\delta_e$ (a unit point mass at some $e \in H$) under the weak-$^*$ topology—for example, $H$ a compact commutative topological semigroup with unit.

For a hypergroup $H$ there exists a continuous map $\lambda: H \times H \longrightarrow M(H)$ defined by $\lambda(x, y) = \delta_x * \delta_y \in M_p(H)$. For $f \in C(H)$, the space of continuous functions on $H$, and $x \in H$ define $R(x)f \in C(H)$, by

$$R(x)f(y) = \int_H f d\lambda(y, x) \qquad (y \in H).$$

If a hypergroup $H$ possesses an invariant measure $m \in M_p(H)$, (that is, $\int_H R(x)f dm = \int_H f dm$, $f \in C(H)$, $x \in H$) and a continuous involution $x \mapsto x'$ $(x \in H)$ such that

$$(1) \qquad \int_H (R(x)f)\bar{g}\, dm = \int_H f(R(x')g)^{-} dm \qquad (f, g \in C(H),\ x \in H),$$

and

$$e \in \operatorname{spt} \lambda(x, x') \qquad (x \in H),$$

then $H$ is called a $^*$-hypergroup ($\operatorname{spt} \mu$ denotes the minimum closed subset of $H$ carrying the measure $\mu$).

A nonzero function $\phi \in C(H)$ is a character if the following formula holds:

$$\phi(x)\phi(y) = \int_H \phi \, d\lambda(x, y) \qquad (x, y \in H).$$

Consequences of these definitions are (1) The space $\hat{H}$ of characters is an orthogonal basis for $L^2(H, dm)$, and (2) $\operatorname{spt} m = H$.

It is not now known whether a $^*$-hypergroup $H$ has the linear span $[\hat{H}]$ of $\hat{H}$ dense in $C(H)$ (in the sup-norm topology). If a $^*$-hypergroup $H$ has the further property that $\hat{H}\hat{H} \subset \operatorname{co}\hat{H}$, the convex hull of $\hat{H}$, then $H$ is said to be a $P_*$-hypergroup; and a fortiori, for compact $P_*$-hypergroups $\hat{H}$ is a topological basis for $C(H)$.

Examples of compact $P_*$-hypergroups include compact abelian groups, the space of conjugacy classes of a compact nonabelian group, and the space of two-sided cosets in certain homogeneous space, for example in $SO(n)/SO(n-1)$ $(n \geqslant 3)$ (see [1, §4]).

*In the sequel, $H$ will always be a compact $P_*$-hypergroup.*

2. **Symmetrization of hypergroups.** The method of symmetrization of a $P_*$-hypergroup was introduced by the authors in [3]. We will in this paper use this construction to produce a denumerable compact $P_*$-hypergroup—a striking contrast to infinite compact groups.

Given a homeomorphism $\tau$ on a compact $P_*$-hypergroup $H$, define $\tau_1$: $C(H) \longrightarrow C(H)$ by $\tau_1 f(x) = f(\tau x)$, $f \in C(H)$, $x \in H$. Let $\tau_1^*$ be the (weak-$^*$ continuous) adjoint of $\tau_1$—that is,

$$\int_H f \, d\tau_1^* \mu = \int_H f \circ \tau_1 \, d\mu \qquad (f \in C(H), \mu \in M(H)).$$

The homeomorphism $\tau$ is called an automorphism if $\tau_1^* \lambda(x, y) = \lambda(\tau x, \tau y)$ $(x, y \in H)$. This implies that, for $\phi \in \hat{H}$, $\phi \circ \tau \in \hat{H}$, and that $\tau(x)' = \tau(x')$ $(x \in H)$.

Let $W$ be a compact group of automorphisms on the compact $P_*$-hypergroup $H$—the topology on $W$ is the pointwise topology from $H$, and the map $(x, \tau) \longmapsto \tau(x)$ of $H \times W \longrightarrow H$ is separately continuous.

2.1. DEFINITION. For $H$ a compact $P_*$-hypergroup and $W$ a compact group, we define the symmetrization operator $\sigma_1$ on $C(H)$ by

$$\sigma_1 f(x) = \int_W f(\tau x) \, dm_W(\tau) \qquad (f \in C(H), x \in H),$$

where $m_W$ denotes the Haar measure on $W$.

We define the compact space $H_W$ by identifying the points of $H$ which are in the same orbit; that is, $H_W = H/\sim$ where $x \sim y$ if and only if there exists $\tau \in W$ such that $\tau x = y$.

2.2. REMARK. In [3, Theorem 4.5] we showed that the space $H_W$ is a compact $P_*$-hypergroup, and that the space $\hat{H}_W$ of characters of $H_W$ is the set $\sigma_1 \hat{H}$ viewed as a subspace of $C(H_W)$.

2.3. EXAMPLES. (1) Let $T$ be the unit circle group $\{e^{i\theta}: 0 \leqslant \theta < 2\pi\}$, and let $W = \{\mathrm{id}, \tau\}$, where $\mathrm{id}(x) = x$ and $\tau(x) = \bar{x}$, $x \in T$. Then $T_W = [0, \pi]$, and the symmetrized characters $\phi \in \hat{T}_W$ are the cosine functions $\cos m\theta$, $\theta \in [0, \pi]$, $m \in Z_+$ (the nonnegative integers).

(2) If in Example 2.3(1), we replace $[0, \pi]$ by $[-1, 1]$ by the transformation $\cos\theta = x$, then $[-1, 1]$ is a $P_*$-hypergroup. Since the Tchebyshev polynomials of the first kind satisfy

$$T_m(x)T_m(y) = \tfrac{1}{2}T_m(xy - \sqrt{(1-x^2)(1-y^2)})$$

$$+ \tfrac{1}{2}T_m(xy + \sqrt{(1-x^2)(1-y^2)}), \qquad x, y \in [-1, 1],\ m \in Z_+,$$

it follows that the hypergroup structure in $[-1, 1]$ is given by

$$\lambda(x, y) = \tfrac{1}{2}\delta(xy - \sqrt{(1-x^2)(1-y^2)})$$

$$+ \tfrac{1}{2}\delta(xy + \sqrt{(1-x^2)(1-y^2)}), \qquad x, y \in [-1, 1],$$

where $\delta(z)$ denotes the unit point mass at $z$.

(3) By letting the permutation group on $N$ letters act on the $N$-fold Cartesian product of a two-point hypergroup, one obtains Krawtchouk polynomials as characters (see [3] for the details).

3. **Symmetrization of the $p$-adic integers.** Fix a prime $p$ and let $\Delta_p$ denote the ring of $p$-adic integers. Each $x \in \Delta_p$ has a unique expansion $x = x_0 + x_1 p + \cdots + x_n p^n + \cdots$ where $x_j = 0, 1, \cdots, p-1$ for $j \geqslant 0$. Let $W$ denote the group of units, that is, $\{x \in \Delta_p: x_0 \neq 0\}$. The norm $|\cdot|_p$ on $\Delta_p$ is defined by $|0|_p = 0$, $|x|_p = p^{-k}$ where $k = \min\{j: x_j \neq 0\}$ for $x \neq 0$. Then $W = \{x: |x|_p = 1\}$ and $x = wy$ for some $w \in W$ if and only if $|x|_p = |y|_p$. Thus $\Delta_p$ is the union of countably many $W$-orbits $\{\xi_j: j = 0, 1, \cdots, \infty\}$ where $\xi_\infty = \{0\}$ and $\xi_j = \{x: |x|_p = p^{-j}\}$ for $j = 0, 1, \cdots$. The space of orbits is homeomorphic to $Z_+^*$, the one-point compactification of $Z_+$, the nonnegative integers.

To preserve the notation from §2, we will use $H$ for $\Delta_p$. The above remarks show the following (note that the points of $H_W$ are the orbits $\xi_j$).

PROPOSITION 3.1. *For $H = \Delta_p$, and $W$ the group of units in $\Delta_p$, the symmetrized $P_*$-hypergroup $H_W$ is homeomorphic to $Z_+^*$.*

We compute the invariant measure $m_{H_W}$ which is nothing but the symmetrization of the Haar measure of $\Delta_p$ (see [3, Corollary 3.10]). For convenience we write $m_k$ for $m_{H_W}(\{\xi_k\})$, $k \in Z_+^*$.

PROPOSITION 3.2. *For $0 \leqslant k < \infty$, $m_k = (1/p)^k(1 - 1/p)$ and $m_\infty = 0$.*

PROOF. Write $m_H$ for the Haar measure of $\Delta_p$. Then

$$m_k = m_H(\xi_k) = m_H\{x: |x|_p \leqslant p^{-k}\} - m_H\{x: |x|_p \leqslant p^{-k-1}\} = p^{-k} - p^{-k-1},$$

as claimed. (Note that $\{|x|_p \leqslant p^{-k-1}\}$ is a subgroup of index $p$ in $\{|x|_p \leqslant p^{-k}\}$ and proceed by induction.) Finally $m_\infty = m_H(\{0\}) = 0$. □

Recall from Remark 2.2 that $\hat{H}_W = \sigma_1 \hat{H}$, the symmetrized characters of $\Delta_p$. The characters of $\hat{H}_W$ will be interpreted as functions on $Z_+^*$ and the general theory [1, Theorem 3.5] shows that they form an orthogonal system relative to the measure $\{m_k\}$ introduced above. We proceed to the direct calculation of the characters. Essentially this depends on the method of Gaussian sums [4, Chapter 20].

DEFINITION 3.3. For each $n = 1, 2, \cdots$ define $\pi_n: \Delta_p \longrightarrow Z(p^n)$ by $\pi_n x = \Sigma_{j=0}^{n-1} x_j p^j \pmod{p^n}$. Then $\pi_n$ is a ring homomorphism, which maps $W$ onto the units in $Z(p^n)$.

Each additive character of $Z(p^n)$ is of the form $\phi_j: x \longmapsto e(xj/p^n)$, $j = 0, 1, \cdots, p^n - 1$. (We will use the notation $e(y) = \exp(2\pi i y)$ in this section.) Each character of $\Delta_p$ is of the form $x \longmapsto \phi_j(\pi_n x)$ some $n, j$ (that is, $\hat{\Delta}_p$ is the injective limit of $Z(p^n)$, called $Z(p^\infty)$). To see the action of $W$ on such a character, note that $\phi_j(\pi_n(wx)) = \phi_j((\pi_n w)(\pi_n x))$, thus for a given character, integration over $W$ can be replaced by a finite sum over the units of $Z(p^n)$ (some $n$).

LEMMA 3.4. *Let $\phi \in \hat{\Delta}_p$. Choose an integer $n$ so that $\phi$ corresponds to some $\phi_j \in Z(p^n)^\wedge$, then the symmetrization of $\phi$ is given by*

$$\sigma_1\phi(x) = \frac{1}{p^{n-1}(p-1)} \sum_{w=1}^{p^n-1}{}' \phi_j(w\pi_n x)$$

*(where $\Sigma'$ indicates summation over $w$ with the* g.c.d.$(w, p) = 1$, *that is, a reduced residue class system). Clearly $\sigma_1\phi(0) = 1$.*

Each $x \neq 0$ can be uniquely written as $x = wp^m$, some $w \in W$, $m = 0, 1, \cdots$, and $\sigma_1\phi(x) = \sigma_1\phi(p^m)$, for $\phi \in \hat{\Delta}_p$, so it suffices to calculate $\sigma_1\phi(p^m)$. Let $\phi_0$ denote the trivial character $\equiv 1$, then $\sigma_1\phi_0 \equiv 1$. Suppose now $\phi \neq \phi_0$; then there exist integers $n$ and $j$ such that $1 \leqslant j \leqslant p^n - 1$ and $\phi(x) = \phi_j(\pi_n x)$ $(x \in \Delta_p)$. Further $\pi_n p^m = 0$ for $m \geqslant n$, $\pi_n 0 = 0$, thus $\sigma_1\phi_j(p^m) = 1$ for $m \geqslant n$. Since $\pi_n p^m = p^m$ for $0 \leqslant m \leqslant n-1$ we must evaluate

$$\sigma_1\phi(p^m) = \frac{1}{p^{n-1}(p-1)} \sum_{w=1}^{p^n-1}{}' e(wjp^{m-n}) \quad \text{for } 0 \leqslant m \leqslant n-1.$$

But $j$ has a unique expression $j = p^k w_1$ some $k$ with $0 \leqslant k \leqslant n-1$ and $w_1$ a unit in $Z(p^n)$. Thus we want $\Sigma'_w e(wp^{m+k-n})$ for $0 \leqslant m, k \leqslant n-1$.

LEMMA 3.5. *For* $n \geqslant 1$ *let* $F_n(z) = \Sigma'^{p^n-1}_{w=1} z^w$ $(z \in \mathbf{C})$, *then*

$$F_n(z) = z(z^{p-1} - 1)(z^{p^n} - 1)/(z - 1)(z^p - 1).$$

*If* $z^p = 1$ *and* $z \neq 1$ *then* $F_n(z) = -p^{n-1}$, *and* $F_n(1) = p^{n-1}(p-1)$.

PROOF. Proceed by induction. It suffices to prove the formula for $|z| < 1$ (so $z^m \neq 1$ for $m = 0, 1, 2, \cdots$). First $F_1(z) = z + z^2 + \cdots + z^{p-1} = z(z^{p-1} - 1)/(z - 1)$. Further

$$F_{n+1}(z) = F_n(z) + \sum_{k=1}^{p-1} \sum_{w=1}^{p^n-1}{}' z^{kp^n+w}$$

$$= F_n(z) + \sum_{k=1}^{p-1} z^{kp^n} F_n(z) = F_n(z)(z^{p^{n+1}} - 1)/(z^{p^n} - 1),$$

which completes the induction. The special values for $z^p = 1$ are easily obtained. □

We return to the main computation. Lemma 3.4 shows that $\sigma_1\phi_j(p^m) = (p^{n-1}(p-1))^{-1}F_n(e(p^{m+k-n}))$ (where $j = w_1p^k$), and Lemma 3.5 gives the values

$$\begin{array}{rl} 1 & \text{if } m + k \geqslant n, \\ -1/(p-1) & \text{if } m + k = n - 1, \\ 0 & \text{if } 0 \leqslant m + k \leqslant n - 2. \end{array}$$

Note that the values depend only on $n - k$, for each given $m$. Thus the symmetrized characters have been determined, and they are naturally indexed by a single integer.

DEFINITION 3.6. For $n = 0, 1, 2, \cdots$ define a function $\chi_n$ on $Z_+^*$ by

$$\chi_n(m) = \begin{cases} 1, & m \geqslant n \text{ or } m = \infty, \\ -1/(p-1), & m = n - 1, \\ 0, & m \leqslant n - 2. \end{cases}$$

THEOREM 3.7. *The functions* $\{\chi_n: n = 0, 1, \cdots\}$ *are the characters of* $H_W$ *under the identification of the orbit* $\xi_j$ *with* $j$ $(j = 0, 1, \cdots, \infty)$.

PROOF. The trivial character $\phi_0 \equiv 1$ symmetrizes (trivially) to $\chi_0$. If $\phi \in \hat{\Delta}_p$, $\phi \neq \phi_0$, then write $\phi(x) = e(jx/p^n)$ some $n = 1, \cdots$ and $1 \leqslant j \leqslant p^n - 1$. Put $j = p^k w_1$ and $x = p^m w$ $(w, w_1 \in W)$ then $\hat{\sigma}_1\phi(x) = \chi_{n-k}(m)$, by the above calculations. Also $\hat{\sigma}_1\phi(0) = \chi_{n-k}(\infty)$. □

It is easy to compute that $\Sigma_{k=0}^{\infty} m_k |\chi_n(k)|^2 = (p^{n-1}(p-1))^{-1}$, for $n \geqslant 1$, and $\Sigma_{k=0}^{\infty} m_k |\chi_0(k)|^2 = 1$.

Having determined the characters of $H_W$ we will now find the structural measures $\lambda(m, n)$, the convolution of $\delta_m$ with $\delta_n$, $m, n \in Z_+$. Since $\infty$ is the identity (corresponding to $0 \in \Delta_p$) we see that $\lambda(m, \infty) = \delta_m$ for each $m \in Z_+$. The others will be found using the identities $\chi_k(m)\chi_k(n) = \int_{H_W} \chi_k d\lambda(m, n)$ for $k = 0, 1, 2, \cdots$.

THEOREM 3.8. *Let* $\lambda$ *be as in* §1. *Then for* $m \neq n$,

$$\lambda(n, m) = \delta(\min(n, m)) \qquad (n, m \in Z_+), \quad and$$

$$\lambda(n, n)(t) = \begin{cases} 0, & t < n, \\ \dfrac{p-2}{p-1}, & t = n, \\ 1/p^k, & t = n + k > n \quad (n, t \in Z_+). \end{cases}$$

PROOF. Let $n, m \in Z_+$ with $n < m$. Since for all characters $\chi_l$ $(l \in Z_+)$ one has

$$\chi_l(n) = \chi_l(n)\chi_l(m) = \int_{H_W} \chi_l d\lambda(n, m),$$

it follows that $\lambda(n, m) = \delta(n)$.

Now let $n \in Z_+$ and note that $\lambda(n, n)(t) = 0$ for $t < n$ since $1 = \chi_n(n)\chi_n(n) = \int_{H_W} \chi_n d\lambda(n, n)$, and so

$$\operatorname{spt} \lambda(n, n) \subset \{l \in Z_+: \chi_n = 1\} = \{l \in Z_+: l \geq n\} \cup \{\infty\}.$$

For notational convenience, let $A_m = \lambda(n, n)(m)$ and so $A_m = 0$ for $m < n$.

Since $\chi_{n+1}(n)\chi_{n+1}(n) = \int_H \chi_{n+1} d\lambda(n, n)$, one has

$$\left(\frac{1}{1-p}\right)^2 = \left(\frac{1}{1-p}\right) A_n + (1 - A_n) = \left(\frac{p}{1-p}\right) A_n + 1,$$

$$A_n = (p-2)/(p-1).$$

Let $m > n + 1$, then

$$0 = \chi_m(n)\chi_m(n) = \int_{H_W} \chi_m d\lambda(n, n)$$

$$= \frac{1}{1-p} A_{m-1} + \left(1 - \sum_{k=n}^{m-1} A_k\right),$$

$$0 = A_{m-1} + (1-p) - (1-p)A_{m-1} - (1-p)\sum_{k=n}^{m-2} A_k,$$

$$pA_{m-1} = (p-1)\left(1 - \sum_{k=n}^{m-2} A_k\right).$$

We compute now (with $m = n + 2$),

$$A_{n+1} = \frac{(p-1)}{p}\left(1 - \frac{p-2}{p-1}\right) = \frac{1}{p}.$$

Suppose $A_{n+l} = (1/p)^l$; then

$$A_{n+l+1} = \left(\frac{p-1}{p}\right)\left(1 - \frac{p-2}{p-1} - \sum_{k=1}^{l} \frac{1}{p^k}\right) = \left(\frac{1}{p}\right)^{l+1}. \quad \square$$

Our next goal will be to compute the multiplication table for the characters $\{\chi_l\colon l = 0, 1, \cdots\}$. Observe that $\chi_n\chi_m = \chi_m$ for $n < m$. Let $n \geqslant 1$ and write $\chi_n^2 = \Sigma_{j=0}^n c_j\chi_j$. Thus for $1 \leqslant l \leqslant n$,

$$\int_{H_W} \chi_n^2\chi_l\,dm = \sum c_j \int_{H_W} \chi_j\chi_l\,dm = c_l \int_{H_W} \chi_l^2\,dm = c_l p^{1-l}(p-1)^{-1};$$

and for $l = 0$, we have $1/p^{n-1}(p-1) = \int_{H_W}\chi_n^2\chi_0\,dm = c_0$.

Now for $1 \leqslant l < n$,

$$\begin{aligned} c_l &= p^{l-1}(p-1)\int_{H_W} \chi_n^2\chi_l\,dm \\ &= p^{l-1}(p-1)\left(\left(\frac{1}{1-p}\right)^2 m_{n-1} + \sum_{k=n}^{\infty} m_k\right) \\ &= p^{l-1}(p-1)\left(\left(\frac{1}{1-p}\right)^2\left(1-\frac{1}{p}\right)\left(\frac{1}{p}\right)^{n-1} + \left(\frac{1}{p}\right)^n\right) \\ &= p^l/p^n = p^{l-n}. \end{aligned}$$

For $l = n$,

$$\begin{aligned} c_n &= p^{l-1}(p-1)\int_{H_W} \chi_n^2\chi_n\,dm \\ &= p^{l-1}(p-1)\left(\left(\frac{1}{1-p}\right)^3 m_{n-1} + \sum_{k=n}^{\infty} m_k\right) \\ &= p^{l-1}(p-1)\left(\left(\frac{1}{1-p}\right)^3\left(1-\frac{1}{p}\right)\left(\frac{1}{p}\right)^{n-1} + \left(\frac{1}{p}\right)^n\right) \\ &= (p-2)/(p-1). \end{aligned}$$

We have thus shown the following:

THEOREM 3.9. *For* $n \geqslant 1$,

$$\chi_n^2 = \frac{1}{p^{n-1}(p-1)}\chi_0 + \sum_{k=1}^{n-1} p^{k-n}\chi_k + \frac{p-2}{p-1}\chi_n.$$

**4. A family of countable compact $P_*$-hypergroups.** Motivated by the results of §3, we will in this section show how to construct for any $a$, $0 < a \leqslant \frac{1}{2}$, a compact countable $P_*$-hypergroup. For $p$ prime and $a = 1/p$ the example agrees with the hypergroup $H_W$ constructed in §3.

Let $a$ be such that $0 < a \leqslant \frac{1}{2}$ and define $H_a$ to be the compact space $Z_+^*$. Define the measure $m$ on $H_a$ by

$$m(k) = \begin{cases} (1-a)a^k, & k \neq \infty, \\ 0, & k = \infty. \end{cases}$$

For each $n \in Z_+$, define

$$\chi_n(k) = \begin{cases} 0, & k < n-1, \\ a/(a-1), & k = n-1, \\ 1, & k \geqslant n \text{ or } k = \infty. \end{cases}$$

For $n, m \in Z_+$ with $n \neq m$, define $\lambda(n, m) = \delta(\min(n, m))$, and for $n = m$ let

$$\lambda(n, n)(t) = \begin{cases} 0, & t < n, \\ \dfrac{1-2a}{1-a}, & t = n, \\ a^k, & t = n + k > n. \end{cases}$$

THEOREM 4.1. *The space $H_a$ $(0 < a \leqslant \frac{1}{2})$ is a compact $P_*$-hypergroup with characters $\{\chi_l: l = 0, 1, \cdots\}$, invariant measure $m$, and the trivial involution $x' = x$. Also*

$$\int_{H_a} \chi_n^2 \, dm = \begin{cases} 1, & n = 0, \\ a^n/(1-a), & n \geqslant 1, \end{cases}$$

*and for $n \in Z_+$ we have*

$$\chi_n^2 = \frac{a^n}{1-a}\chi_0 + \sum_{k=1}^{n-1} a^{n-k}\chi_k + \frac{1-2a}{1-a}\chi_n.$$

PROOF. That $H_a$ is a $P_*$-hypergroup is straightforward. The other computations are just like those in §3. □

REMARK 4.2. The compact $P_*$-hypergroup $H_a$ $(0 < a \leqslant \frac{1}{2})$ has a neighborhood basis of open subhypergroups at the identity $e = \infty$; namely, the sets $\{n: n \geqslant m\} \cup \{\infty\}$ for each $m \in Z_+$, whose annihilators in $\hat{H}$ are $\{\chi_0, \chi_1, \cdots, \chi_m\}$.

5. **Homomorphisms between hypergroups.** In this section, $a, b$ are such that $0 < a, b \leqslant \frac{1}{2}$ and $H_a, H_b$ are defined as in §4. The characters of $H_a, H_b$ will be denoted by $\chi_l^{(a)}, \chi_l^{(b)}$ respectively, $l \in Z_+$. The invariant measure on $H_a$, $H_b$ will be denoted by $m^{(a)}, m^{(b)}$ respectively.

DEFINITION 5.1. The space $P(H_a)$ is the collection of continuous functions $f$ on $H_a$ which can be expressed as a linear combination of the characters of $H_a$ with summable positive coefficients. We call $f \in P(H_a)$ a positive-definite function on $H_a$.

Since $H_a$ and $H_b$ are both isomorphic to $Z_+^*$ there exists a canonical map from $H_a \longrightarrow H_b$. Let $\pi$ be the induced map of $C(H_b) \longrightarrow C(H_a)$.

THEOREM 5.2. *Let* $0 < b \leqslant a \leqslant \frac{1}{2}$. *Then* $\pi P(H_b) \subset P(H_a)$.

PROOF. For $l \in Z_+$ we will show that $\pi\chi_l^{(b)} \in P(H_a)$. Write $\pi\chi_l^{(b)}$ as $\Sigma_{j=0}^{l} c_j \chi_j^{(a)}$. The functions on $Z_+^*$ which are constant on $[l, \infty]$ are linear combinations of $\{\chi_j\colon 0 \leqslant j \leqslant l\}$. Consider

$$\begin{aligned} c_0 &= c_0 \int_{H_a} \chi_0^{(a)} \chi_0^{(a)}\, dm^{(a)} = \int_{H_a} \left( \sum_{j=0}^{l} c_j \chi_j^{(a)} \right) \chi_0 \, dm^{(a)} \\ &= \int_{H_a} (\pi\chi_l^{(b)}) \chi_0 \, dm^{(a)} = \frac{b}{b-1} m_{l-1}^{(a)} + \sum_{k=l}^{\infty} m_k^{(a)} \\ &= \frac{b}{b-1}(1-a)a^{l-1} + a^l = a^{l-1}\left(\frac{a-b}{1-b}\right) \geqslant 0. \end{aligned}$$

Similarly for $c_l$ we have

$$\begin{aligned} c_l \int_{H_a} \chi_l^{(a)} \chi_l^{(a)}\, dm^a &= \left(\frac{b}{b-1}\right)\left(\frac{a}{a-1}\right) m_{l-1}^{(a)} + \sum_{k=l} m_k^{(a)} \\ &= \left(\frac{b}{b-1}\right)\left(\frac{a}{a-1}\right)(1-a)a^{l-1} + a^l \\ &= a^{l-1}\left(\frac{a}{1-b}\right), \end{aligned}$$

and so $c_l = (1-a)/(1-b) > 0$.

For $1 \leqslant j \leqslant l-1$, we have

$$c_j \int_{H_a} \chi_j^{(a)} \chi_j^{(a)}\, dm^{(a)} = \left(\frac{b}{b-1}\right) m_{l-1}^{(a)} + \sum_{k=l}^{\infty} m_k^{(a)} = a^{l-1}\left(\frac{a-b}{1-b}\right),$$

and so $c_j = (1-a)(a-b)a^{l-1-j}/(1-b) \geqslant 0$. $\square$

COROLLARY 5.3. *For* $0 < b \leqslant a \leqslant \frac{1}{2}$ *and* $l \geqslant 1$,

$$\pi\chi_l^{(b)} = \frac{a^{l-1}(a-b)}{(1-b)}\chi_0^{(a)} + \sum_{j=1}^{l-1} \frac{(1-a)(a-b)a^{l-1-j}}{(1-b)}\chi_j^{(a)} + \frac{(1-a)}{(1-b)}\chi_l^{(a)}.$$

Since $\hat{H}_a$, $\hat{H}_b$ are both isomorphic to $Z_+$, there exists a canonical map $\rho$:

$\hat{H}_a \longrightarrow \hat{H}_b$. We wish to investigate when $\rho$ extends to a positive map of $C(H_a) \longrightarrow C(H_b)$ (that is, $f \geq 0$ implies $\rho f \geq 0$).

THEOREM 5.4. *For* $0 < b \leq a \leq ½$ *and* $l \in Z_+$, *there exists* $\mu_l \in M_p(H_a)$ *such that* $\chi_n^{(b)}(l) = \int_{H_a} \chi_n^{(a)} d\mu_l$ $(n \in Z_+)$.

PROOF. Put

$$\mu_l = \frac{1-a}{1-b}\delta_l^{(a)} + \sum_{s=1}^{\infty} \frac{(1-a)(a-b)}{(1-b)} a^{s-1}\, \delta_{l+s}.$$

Since $|a| \leq ½$, $\mu_l \in M(H_a)$. Since $a \geq b$, $\mu_l \geq 0$. To see that $\mu_l \in M_p(H_a)$ consider

$$\|\mu_l\| = \int_{H_a} 1\, d\mu_l = \frac{1-a}{1-b} + \sum_{s=1}^{\infty} \frac{(1-a)(a-b)}{(1-b)} a^{s-1} = 1.$$

For $n = l + 1$,

$$\int_{H_a} \chi_n^{(a)} d\mu_l = \frac{(1-a)a}{(1-b)(a-1)} + \sum_{s=1}^{\infty} \frac{(1-a)(a-b)}{(1-b)} a^{s-1}$$

$$= \frac{b}{b-1} = \chi_n^{(b)}(l).$$

For $n \leq l$,

$$\int_{H_a} \chi_n^{(a)}\, d\mu_l = \frac{1-a}{1-b} + \sum_{a=1}^{\infty} \frac{(1-a)(a-b)}{(1-b)} a^{s-1} = 1 = \chi_n^{(b)}(l).$$

For $n > l + 1$,

$$\int_{H_a} \chi_n^{(a)} d\mu_l = \frac{(1-a)(a-b)}{(1-b)} a^{n-2-l}\left(\frac{a}{a-1}\right) + \sum_{s=n-l}^{\infty} \frac{(1-a)(a-b)}{(1-b)} a^{s-1}$$

$$= \frac{(1-a)(a-b)}{(1-b)} a^{n-2-l}\left(\frac{a}{a-1} + \frac{a}{1-a}\right) = 0. \quad \square$$

COROLLARY 5.5. *For* $0 < b \leq a \leq ½$, *the map* $\rho: \hat{H}_a \longrightarrow \hat{H}_b$ *extends to a positive map from* $C(H_a) \longrightarrow C(H_b)$.

PROOF. Extend $\rho$ to the linear span $[\hat{H}_a]$ by $\rho(\Sigma_{i=1}^n c_i \chi_i^{(a)}) = \Sigma_{i=1}^n c_i \rho \chi_i^{(a)}$ which is well defined since the characters are linearly independent.

Since $[\hat{H}_a]$ is dense in $C(H_a)$ we need only show that $\rho: [\hat{H}_a] \longrightarrow [\hat{H}_b]$ is bounded and then extend by uniform continuity. Let $f = \Sigma_{i=1}^n c_i \chi_i^{(a)} \in [\hat{H}_a]$, $f \geq 0$, and $l \in Z_+ \subset H_b$; then

$$\rho f(l) = \sum_{i=1}^{n} c_i \rho \chi_i^{(a)}(l) = \sum_{i=1}^{n} c_i \chi_i^{(b)}(l)$$

$$= \sum_{i=1}^{n} c_i \int_{H_a} \chi_i^{(a)} d\mu_l = \int_{H_a} \sum_{i=1}^{n} c_i \chi_i^{(a)} d\mu_l$$

$$= \int_{H_a} f d\mu_l \geqslant 0.$$

Also $\|\rho f\|_\infty \leqslant \|f\|_\infty$, and $\|\rho\| \leqslant 1$. $\square$

6. **Partial summation kernels.** In this section we retain the notation from §§4 and 5. Let $0 < a \leqslant ½$ and $H_a$ the associated compact $P_*$-hypergroup isomorphic to $Z_+^*$.

DEFINITION 6.1. For $f \in C(H_a)$, the Fourier series of $f$ is given by

$$f \sim \hat{f}_0 \chi_0 + \sum_{n=1}^{\infty} (1-a)a^{-n} \hat{f}_n \chi_n,$$

where $\hat{f}_n = \int_{H_a} f \chi_n \, dm$, $n \in Z_+$. The partial summation kernel $K_n$, $n \in Z_+$, is given by

$$K_n = \chi_0 + \sum_{j=1}^{n} (1-a)a^{-j} \chi_j,$$

and so $\int_H K_n \, dm = 1$ (the weights are the $L^2$-norms of $\chi$; see Theorem 4.1).

THEOREM 6.2. *Let* $m, n \in Z_+$; *then*

$$K_n(m) = \begin{cases} 0, & m < n, \\ a^{-n}, & m \geqslant n. \end{cases}$$

PROOF. For $m \geqslant n$,

$$K_n(m) = \chi_0(m) + \sum_{j=1}^{n} (1-a)a^{-j} \chi_j(m) = 1 + \sum_{j=1}^{n} (1-a)a^{-j} = a^{-n}.$$

For $0 < m < n$,

$$K_n(m) = 1 + \sum_{j=1}^{m} (1-a)a^{-j} + (1-a)a^{-(m+1)} a/(a-1)$$

$$= 1 - (1 - 1/a)^m - a^{-m} = 0.$$

Finally,

$$K_n(0) = 1 + (1-a)a^{-1}\chi_1(0) = 1 + (1-a)a^{-1} a/(a-1) = 0. \quad \square$$

THEOREM 6.3. *Let* $f \in C(H_a)$; *then* $K_n * f \xrightarrow{n} f$ *uniformly as* $n \longrightarrow \infty$.

PROOF. Let $\epsilon > 0$. Since the map $x \longmapsto R(x)f$ is uniformly continuous

[1, Theorem 1.10], there exists a neighborhood $V$ of $e = \infty$ such that for $y \in V$, $\|R(y)f - f\|_\infty \leqslant \epsilon$. By Theorem 6.2, there exists $N$ such that, for $n \geqslant N$, $\operatorname{spt} K_n \subset V$. By [1, Proposition 3.4], for $x \in H$,

$$K_n * f(x) = \int_{H_a} R(x)f(y')K_n(y)\,dm(y);$$

but $y = y'$ in $H_a$ and $R(x)f(y) = R(y)f(x)$. Thus

$$|K_n * f(x) - f(x)| = \left|\int_{H_a} (R(y)f(x) - f(x))K_n(y)dm(y)\right|$$
$$\leqslant \|R(y)f - f\|_\infty \leqslant \epsilon. \quad \square$$

DEFINITION 6.4. For $r$ with $0 \leqslant r < 1$, we define the Poisson sum $P_r$ by

$$P_r = \chi_0 + \sum_{n=1}^{\infty} (1 - a)a^{-n}r^n\chi_n,$$

a pointwise finite sum.

THEOREM 6.5. *For $r$ with $0 \leqslant r < 1$, $P_r(0) = 1 - r$; and for $k \in Z_+$,*

$$P_r(k) = (1 - r)\left(1 - \left(\frac{r}{a}\right)^{k+1}\right)\bigg/\left(1 - \left(\frac{r}{a}\right)\right) > 0.$$

PROOF. $P_r(0) = 1 + (1 - a)a^{-1}r\chi_1(0) = 1 + (1 - a)a^{-1}ra/(a - 1) = 1 - r$. For $k \in Z_+$,

$$P_r(k) = 1 + \frac{(1 - a)}{a}r + \frac{(1 - a)}{a^2}r^2 + \cdots + \frac{(1 - a)}{a^k}r^k + \frac{(1 - a)}{a^{k+1}}r^{k+1}\frac{a}{(a - 1)}$$

$$= 1 - \frac{r^{k+1}}{a^k} + (1 - a)\frac{(r/a) - (r/a)^{k+1}}{1 - r/a}$$

$$= 1 - \frac{r^{k+1}}{a^k} + \frac{(1 - a)}{(a - r)}(r - r^{k+1}a^{-k})$$

$$= (a - n)(a - ar + a^{-k}r^{k+2} - a^{-k}r^{k+1})$$

$$= \frac{(1 - r)}{(a - r)}(a - a^{-k}r^{k+1}) = \frac{a(1 - r)}{(a - r)}(1 - (r/a)^k)$$

$$= (1 - r)\frac{1 - (r/a)^k}{1 - r/a} > 0. \quad \square$$

REMARK 6.6. Note, for $0 \leqslant r < 1$, that $P_r > 0$, $\int_H P_r\,dm = 1$ and, for any neighborhood $V$ of $\infty \in H$,

$$\sup\{|P_r(k)|; k \notin V\} \longrightarrow 0 \text{ as } r \longrightarrow 1.$$

By a standard argument, $P_r * f \longrightarrow f$ uniformly as $r \longrightarrow 1$ $(f \in C(H))$.

## CHAPTER II

7. **Characterizations of $F^{-1}l^p(\hat{H})$.** We keep the notation developed in §4. Let $a$ be such that

$$0 < a \leqslant ½, \quad H = Z_+^*, \quad m(k) = (1-a)a^k \quad (k \in Z_+), \quad m(\infty) = 0,$$

and

$$\chi_n(k) = \begin{cases} 0, & k < n-1, \\ a/(a-1), & k = n-1, \\ 1, & k \geqslant n, \end{cases} \qquad (n, k \in Z_+).$$

DEFINITION 7.1. Let $p \geqslant 1$. For a function $f$ on $H$ define

$$\|f\|_p = \left(\sum_{k=0}^{\infty} |f(k)|^p (1-a)a^k\right)^{1/p},$$

and $L^p(H, dm)$ to be the space of all functions $f$ with $\|f\|_p < \infty$.

DEFINITION 7.2. For $k \in Z_+$, define

$$c(k) = c(\chi_k) = \begin{cases} 1, & k = 0, \\ (1-a)/a^k, & k \geqslant 1 \end{cases}$$

(recall Theorem 4.1).

DEFINITION 7.3. Let $p \geqslant 1$. For a function $f$ on $\hat{H}$, define

$$\|f\|_p = \left(\sum_{k=0}^{\infty} |f(k)|^p c(\chi_k)\right)^{1/p},$$

and $l^p(\hat{H})$ to be the space of such functions $f$ with $\|f\|_p < \infty$.

Note that $l^p(\hat{H}) \subset c_0(\hat{H})$.

DEFINITION 7.4. For $f \in L^1(H)$, define $\hat{f} \in c_0(H)$ by $\hat{f}(n) = \hat{f}_n = \int_H f\chi_n\, dm$. The map $f \mapsto \hat{f}$ is denoted by $F$.

DEFINITION 7.5. For $f \in l^1(\hat{H})$ define the function $F^{-1}f \in C(H)$ by

$$F^{-1}f = \sum_{k=0}^{\infty} c(\chi_k) f(k) \chi_k.$$

The space $F^{-1}l^1(\hat{H})$ is denoted by $A(H)$. For $1 < p \leqslant 2$, define $F^{-1}l^p(\hat{H})$ to be the subspace of $L^1(H)$ of those functions $f$ such that $Ff \in l^p(\hat{H})$.

REMARK 7.6. For $p$ a prime and $\Delta_p$ the space of $p$-adic integers, let $G_n = \{x \in \Delta_p : x_l = 0 \text{ for } 0 \leqslant l < n\}$. René Spector [5] defined a function $f$ on $\Delta_p$ to be radial if it is constant on each subset of $\Delta_p$ of the form $G_n \backslash G_{n+1}$, called a corona. The space of continuous radial functions on $\Delta_p$ is isomorphic to $C(H)$ (with $a = 1/p$). Spector has defined and characterized the

Fourier transforms of radial functions on $\Delta_p$. We will now state the analogous results for hypergroups. The reader is referred to Spector [5] for the unfortunately tedious proofs. We will however give straightforward proofs in §9 for the case $p = 1$.

PROPOSITION 7.7. *Let* $f \in L^1(H)$; *then* $f(n+1) - f(n) = a^{-(n+1)}(\hat{f}_{n+1} - \hat{f}_{n+2})$, $n \in Z_+$, *and* $f(0) = \hat{f}_0 - \hat{f}_1$.

PROOF (See Spector [5, p. 64]). For $n \geqslant 0$,

$$\begin{aligned}\hat{f}_{n+1} &= \int_H f\chi_{n+1}\,dm \\ &= f(n)(a/(a-1))(1-a)a^n + \sum_{k=n+1}^{\infty} f(k)a^k(1-a) \\ &= -f(n)a^{n+1} + \sum_{k=n+1}^{\infty} f(k)a^k(1-a).\end{aligned}$$

Similarly,

$$\hat{f}_{n+2} = -f(n+1)a^{n+2} + \sum_{k=n+2}^{\infty} f(k)a^k(1-a).$$

Subtracting yields

$$\begin{aligned}\hat{f}_{n+1} - \hat{f}_{n+2} &= -f(n)a^{n+1} + f(n+1)(a^{n+1}(1-a) + a^{n+2}) \\ &= a^{n+1}(f(n+1) - f(n)). \quad \square\end{aligned}$$

THEOREM 7.8. *For* $1 \leqslant p \leqslant 2$, *and* $f \in L^1(H)$, $f \in F^{-1}l^p(\hat{H})$ *if and only if* $\sum_{k=1}^{\infty} |f(k) - f(k-1)|^p a^{k(p-1)} < \infty$.

THEOREM 7.9. *For* $1 < p \leqslant 2$ *and* $f \in L^1(H)$, $f \in F^{-1}l^p(\hat{H})$ *implies* $\sum_{k=0}^{\infty} a^{k(p-1)}|f(k)|^p < \infty$. *The converse holds for* $1 \leqslant p \leqslant 2$.

REMARK 7.10. Let $f$ be a function on $H$ with either

$$\sum_{k=1}^{\infty} |f(k) - f(k-1)|^p a^{k(p-1)} < \infty \qquad (1 \leqslant p \leqslant 2),$$

or

$$\sum_{k=0}^{\infty} |f(k)|^p a^{k(p-1)} < \infty \qquad (1 < p \leqslant 2).$$

Then $f \in L^1(H)$. To see this, note that these two conditions both define norms on the trigonometric polynomials which are equivalent to the norm given from $l^p(\hat{H})$; and that the trigonometric polynomials are dense in $l^p(\hat{H})$ as well as the weighted $L^p$-space of functions defined by these two conditions.

8. **Characterizations of** $\digamma L^p(H)$. The space $\hat{H} = Z_+$ is a $*$-hypergroup with the invariant measure being $c$ (Definition 7.2) and conjugation being the involution. Using Proposition 7.7, we now characterize $\digamma L^p(H)$ $(1 \leqslant p \leqslant 2)$.

For $f \in \digamma L^p(H)$, let $\hat{f} \in L^p(H)$ be the unique function with $\digamma(\hat{f}) = f$.

THEOREM 8.1. *For* $1 \leqslant p \leqslant 2$ *and* $f \in \digamma L^1(H)$, $f \in \digamma L^p(H)$ *if and only if* $\Sigma_{k=0}^{\infty} a^{-k(p-1)} |f(k) - f(k+1)|^p < \infty$.

THEOREM 8.2. *For* $1 < p \leqslant 2$ *and* $f \in \digamma L^1(H)$, $f \in \digamma L^p(H)$ *implies* $\Sigma_{k=0}^{\infty} a^{-k(p-1)} |f(k)|^p < \infty$. *The converse holds for* $1 \leqslant p \leqslant 2$.

REMARK 8.3. Let $f$ be a function on $\hat{H}$ with $f(k) \longrightarrow 0$ as $k \longrightarrow \infty$ and either

$$\sum_{k=1}^{\infty} |f(k) - f(k-1)|^p a^{-k(p-1)} < \infty \qquad (1 \leqslant p \leqslant 2),$$

or

$$\sum_{k=0}^{\infty} |f(k)|^p a^{-k(p-1)} < \infty \qquad (1 < p \leqslant 2).$$

Then $f \in \digamma L^1(H)$. To see this, note that these two conditions both define norms on the trigonometric polynomials which are equivalent to the norms given from $L^p(H)$; and that the trigonometric polynomials are dense in $L^p(H)$ as well as the weighted $L^p$-space of functions vanishing at infinity defined by these two conditions.

9. **A characterization of** $\digamma L^1(H)$. In this section we give a short proof of Theorem 8.1 for the case $p = 1$.

DEFINITION 9.1. For $f$ a function on $\hat{H} = Z_+$ $(0 < a \leqslant ½)$, define $\|f\|_{bv} = \Sigma_{k=0}^{\infty} |f(k) - f(k+1)|$. The space $bv_0(\hat{H})$ is the collection of all $f$ such that $\|f\|_{bv} < \infty$ and which vanish at infinity.

LEMMA 9.2. *Let* $\mu \in M(H)$ $(0 < a \leqslant ½)$, *then* $\mu\{\infty\} = \lim_{k \to \infty} \hat{\mu}(k)$.

PROOF. Write $\mu = \mu_a + \mu_d$ where $\mu_a \in L^1(H)$ and $\mu_d = \mu | \{\infty\}$. Then since $L^1(H)^\wedge \subset c_0(\hat{H})$, $\hat{\mu}_a(k) \longrightarrow 0$ as $k \longrightarrow \infty$. Also $\hat{\mu}_d(k) = \mu\{\infty\}$, $k \geqslant 0$. □

PROPOSITION 9.3. *Let* $\mu \in L^1(H)$. *Then* $\|\hat{\mu}\|_{bv} \leqslant ((1+a)/(1-a)) \|\mu\|$; *and thus* $\digamma L^1(H) \subset bv_0(\hat{H})$.

PROOF. Let $\mu \in L^1(H)$. Write $\mu = \Sigma_{n=0}^{\infty} \mu_n \delta_n$, $\|\mu\| = \Sigma_{n=0}^{\infty} |\mu_n| < \infty$. Now

$$\|\hat{\mu}\|_{bv} = \left\| \sum_{n=0}^{\infty} \mu_n \hat{\delta}_n \right\|_{bv}$$
$$\leq \|\mu\| \sup \{\|\hat{\delta}_n\|_{bv} : n \in Z_+\} \leq \|\mu\|(1 + 2a/(1-a))$$

since

$$\hat{\delta}_n(k) = \int \chi_k d\delta(n) = \chi_k(n) = \begin{cases} 0, & k > n+1, \\ a/(a-1), & k = n+1, \\ 1, & k < n+1. \quad \square \end{cases}$$

THEOREM 9.4. $FL^1(H) = bv_0(\hat{H})$ $(0 < a \leq \frac{1}{2})$. *Also for* $f \in L^1(H)$, $\|f\|_1 \leq \|\hat{f}\|_{bv}$.

PROOF. Let $g \in bv_0(\hat{H})$. Define $g_k \in bv_0(\hat{H})$ $(k \in Z_+)$ by

$$g_k(l) = \begin{cases} 1, & \text{if } l \leq k, \\ 0, & \text{if } l > k. \end{cases}$$

Write $g = \Sigma_{k=0}^{\infty} c_k g_k$ where $c_0 = 0$, $c_n = g(n) - g(n+1)$, $g(n) = \Sigma_{k=n}^{\infty} c_k$, and $\Sigma_{k=0}^{\infty} |c_k| = \|g\|_{bv} < \infty$.

To show $g \in L^1(H)\hat{}$ it suffices to show that $F^{-1}g_k \in L^1(H)$ and $\|F^{-1}g_k\|_1 \leq 1$, $k \in Z_+$.

For this result, recall the definition of the partial summation kernel $K_n$ from §6: $K_n = \chi_0 + \Sigma_{j=1}^{n}(1-a)a^{-j}\chi_j$. Thus $K_n = \Sigma_{j=0}^{\infty} c(\chi_j) g_n \chi_j$, and so $\hat{K}_n = g_n$; equivalently $F^{-1}g_n = K_n \in L^1(H)$. Also recall $\int_H K_n dm = 1$. Hence

$$\|F^{-1}g\|_1 \leq \left\| F^{-1} \sum_{k=0}^{\infty} c_k g_k \right\|_1 \leq \sum_{k=0}^{\infty} |c_k| = \|g\|_{bv}. \quad \square$$

## CHAPTER III

10. **Banach algebra considerations.** Let $0 < a \leq \frac{1}{2}$ and $H = Z_+^*$ be as in Chapter II. The Fourier algebra $A(H)$ of $H$ by Theorem 7.8 is the same as the space $BV(H)$ of functions of bounded variation on $H$. Thus we have:

THEOREM 10.1. *The Fourier algebra* $A(H)$ *of* $H$ $(0 < a \leq \frac{1}{2})$ *is a regular Banach algebra with* $H$ *as its maximal ideal space.*

Since $M(H) \cong L^1(H) \oplus \mathbf{C}$ by the decomposition $\mu = \mu|H\backslash\{\infty\} + \mu|\{\infty\}$, we have:

THEOREM 10.2. *The maximal ideal space of* $M(H)$ $(0 < a \leq \frac{1}{2})$ *is* $\hat{H} \cup \{\chi_\infty\}$ *where* $\chi_\infty(\mu) = \mu\{\infty\}$.

DEFINITION 10.3. A set $E \subset H$ is called a Helson set if $E$ is closed and $A(H)|E = C(E)$. A set $E \subset \hat{H}$ is called a Sidon set if $\mathcal{F}L^1(H)|E = c_0(E)$.

Using Theorem 7.8 and Theorem 9.4 we have:

THEOREM 10.4. *The Helson (Sidon) sets of $H$ $(\hat{H})$ are the finite subsets of $H$ $(\hat{H})$ respectively.*

DEFINITION 10.5. A subset $E \subset H$ is said to be a set of spectral synthesis for $A(H)$ if given $\epsilon > 0$ and $f \in A(H)$ with $f = 0$ on $E$, there exists $g \in A(H)$ with $\|f - g\|_{A(H)} < \epsilon$ and $g = 0$ on a neighborhood of $E$.

Once again since $A(H) = BV(H)$, we have:

THEOREM 10.6. *Every subset $E$ of $H$ is a set of spectral synthesis for $A(H)$.*

DEFINITION 10.7. A function $f$ on $\mathbf{C}$ is said to be Lipschitz provided $|f(x) - f(y)| \leqslant l\,|x - y|$, $l < \infty$ $(x, y \in \mathbf{C})$. A function $g$ on $\mathbf{C}$ is said to operate on $A(H)$ provided given $f \in A(H)$ (with $f(H) \subset$ domain $g$), then $g \circ f \in A(H)$.

THEOREM 10.8. *The functions which operate on $A(H)$ $(0 < a \leqslant \frac{1}{2})$ are the Lipschitz functions.*

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